UiO : Department of Mathematics
University of Oslo

# Strongly 1-Bounded Quantum Group von Neumann Algebras

Quantum Groups Seminar

**Floris Elzinga**

**23rd November 2021**

# Introduction

Contents:

- Free orthogonal quantum groups and quantum automorphism groups
- $\mathrm{II}_1$-factors and free probability
- Strong 1-boundedness (S1B)
- $L^\infty(O_N^+)$ is S1B
- Subfactors
- New result: some quantum automorphism groups are S1B
- Open problems and obstacles

The last part is based on ongoing work with Michael Brannan, Sam Harris, and Makoto Yamashita.

# Free Orthogonal Quantum Groups

**Definition (van Daele–Wang)**

Let $N \geq 2$ and $Q \in \mathrm{GL}_N(\mathbb{C})$ such that $Q\overline{Q} \in \mathbb{C}I_N$. The free orthogonal quantum group $\mathbb{F}O(Q)$ is the compact matrix quantum group given by the C*-algebra

$$C_u(O_F^+) = \left\langle u_{ij} \,\middle|\, 1 \leq i,j \leq N,\ u \text{ unitary},\ Q\overline{u}Q^{-1} = u \right\rangle .$$

- This is of Kac type if and only if $Q = I_N$ or $Q = J_{2M}$ (up to isomorphism), with $J_{2M}$ standard symplectic.
- Denote $O_{I_N}^+ = O_N^+$ and $O_{J_{2M}}^+ = O_{2M}^{J,+}$; the latter is a *graded twist* of the former.
- If one demands that $\overline{u}$ is unitary but not equal to $u$, one obtains $U_N^+$.
- We will focus on the function algebras $L^\infty(O_N^+)$, $L^\infty(O_{2M}^{J,+})$, and $L^\infty(U_N^+)$.

# Quantum Automorphism Groups

- Let $B$ be a finite dimensional C*-algebra and equip it with its *Markov trace*: this is the restriction of the unique trace on $\mathrm{End}(B)$, where $B$ is included via the left regular representation.
- Let $m\colon B \otimes B \to B$ be the multiplication map and $\nu\colon \mathbb{C} \to B$ the unit map.

**Definition (Wang, Banica)**

The quantum automorphism group of $B$ is the compact matrix quantum group given by the C*-algebra

$$C_u(\mathrm{Aut}^+(B)) = \Big\langle u_{ij} \Big| 1 \leq i,j \leq \dim(B),\ u \text{ unitary}, \nu \in \mathrm{Mor}(\mathrm{triv}, u), \\ m \in \mathrm{Mor}(u \boxtimes u, u) \Big\rangle.$$

- In this setting all $\mathrm{Aut}^+(B)$ are of Kac type.
- Important special cases: $\mathrm{Aut}^+(M_N(\mathbb{C}))$ and $\mathrm{Aut}^+(\mathbb{C}^N) = S_N^+$.

# $\mathrm{II}_1$-factors

- A *$II_1$-factor* is an infinite dimensional von Neumann algebra $\mathcal{M}$ admitting a unique faithful normal tracial state $\tau$.
- Examples:
- The *hyperfinite $II_1$-factor* $\mathcal{R} = M_2(\mathbb{C})^{\otimes\infty}$,
- more generally the group von Neumann algebra of a discrete ICC group, such as the free groups $\mathbb{F}_m$ ($\mathcal{R}$ arises for any amenable such group),
- certain compact quantum group function algebras such as
  - $L^\infty(O_N^+)$ when $N \geq 3$ (Vaes–Vergnioux),
  - $L^\infty(U_N^+)$ when $N \geq 2$ (Vaes),
  - $L^\infty(O_{2M}^{J,+})$ when $M \geq 2$,
  - and $L^\infty(\mathrm{Aut}^+(B))$ when $\dim(B) \geq 8$ (Brannan).

# von Neumann Algebraic Properties

$\mathcal{L}\mathbb{F}_m$ and the Kac type free orthogonal quantum group von Neumann algebras $L^\infty(O_N^+)$ and $L^\infty(O_{2M}^{J,+})$ share many von Neumann algebraic properties (and many group-like approximation properties).

Examples:

- Haagerup property and CCAP (Brannan, De Commer–Freslon–Yamashita)
- strongly solid and in particular possess no Cartan subalgebra (Caspers, Fima–Vergnioux, Isono),
- full and hence prime (Vaes–Vergnioux),
- $\{L^\infty(O_N^+)\}$ has FGF-like asymptotics in a strong sense (Banica, Brannan).

The $L^\infty(\mathrm{Aut}^+(B))$ also satisfy many of these properties. Can we distinguish them?

# Free Probability

- Let $(\mathcal{M}, \tau)$ be a $\mathrm{II}_1$-factor.
- Let $X, Y \in \mathcal{M}$ be self-adjoint, then they are free if for all $2k$-tuples $P_1, \ldots, P_{2k}$ of polynomials we have that $\tau(P_1(X) \cdots P_{2k}(Y)) = 0$ if $\tau(P_1(X)) = \ldots = \tau(P_{2k}(Y)) = 0$.
- This extends to $n$-tuples $X_1, \ldots, X_n$ of self-adjoint elements.
- A self-adjoint $S \in \mathcal{M}$ is a semicircular element if its spectral measure wrt $\tau$, is supported on $[-2, 2]$ and is given by the formula

$$\mathrm{d}\mu_S(t) = \frac{1}{2\pi}\sqrt{4 - t^2}\,\mathrm{d}\lambda(t).$$

- $\mathcal{L}\mathbb{F}_m$ can be characterised as the unique von Neumann algebra generated by $m$ free semicircular elements.

# Free Probability II

- Let $X = (X_1, \ldots, X_n)$ and $Y = (Y_1, \ldots, Y_m)$ be self-adjoint tuples in $\mathcal{M}$.
- The relative microstates free entropy $\chi(X : Y)$ is defined as a limit of the logarithmic volume of *relative microstates*.
- It gives rise to the (modified) microstates free entropy dimension

$$\delta_0(X) = n + \limsup_{\varepsilon \downarrow 0} \frac{\chi(X + \varepsilon S : S)}{|\log \varepsilon|} \leq n.$$

- Here $S$ is an $n$-tuple of free semicircular elements, free from $X$.
- It turns out that $\delta_0(S_1, \ldots, S_m) = m$ (recall these generate $\mathcal{L}\mathbb{F}_m$).
- Unknown if $\delta_0$ is an invariant. If it is, that will settle the free group factor isomorphism problem.

# Strong 1-Boundedness

**Definition (Jung)**

$X$ is 1-bounded if for small $\varepsilon$ we have the estimate

$$\chi(X + \varepsilon S : S) \leq (1 - n)|\log \varepsilon| + \text{const}.$$

If additionally an $X_j$ satisfies $\chi(X_j) > -\infty$, $X$ is strongly 1-bounded. $\mathcal{M}$ is called strongly 1-bounded if it admits such a generating tuple.

- $\chi(X_j) > -\infty$ if it has a bounded diffuse spectral measure.
- 1-Boundedness is slightly stronger than $\delta_0(X) \leq 1$.

**Theorem (Jung)**

*If $\mathcal{M}$ is strongly 1-bounded, then all generating tuples $Y$ are 1-bounded.*

- Hence, such $\mathcal{M}$ are not isomorphic to any $\mathcal{L}\mathbb{F}_m$.
- For $\mathrm{II}_1$-factors, property $\Gamma$ implies being S1B.

# Sufficient Condition for $1$-Boundedness

**Theorem (Jung, Shlyakhtenko)**

*Let $(\mathcal{M}, \tau)$ be a $II_1$-factor and $X = (X_1, \ldots, X_n)$ self-adjoints in $\mathcal{M}$. Assume that there is a vector $F$ of polynomial relations such that*

$$F(X) = 0 \text{ and } \det_{\mathrm{FKL}} \left[\partial F(X)^* \partial F(X)\right] \neq 0.$$

*Then it holds that $X$ is $\alpha$-bounded with*

$$\alpha = n - \operatorname{rank} \partial F(X).$$

- Take $F$ to be the defining relations of $O_N^+$.
- Fact: $\operatorname{rank} \partial F(u) = N^2 - 1 + \beta_0^{(2)}(O_N^+) - \beta_1^{(2)}(O_N^+) = N^2 - 1$.
- In this case $\partial F(u)^* \partial F(u)$ is related to the quantum Cayley graph of the discrete dual of $O_N^+$.
- Original setting: sofic groups $\Gamma$ with $\beta_1^{(2)}(\Gamma) = 0$

# $L^\infty(O_N^+)$ is S1B

**Theorem (Brannan–Vergnioux)**

*$L^\infty(O_N^+)$ is strongly 1-bounded.*

- $\delta_0(X)$ actually only depends on the $*$-algebra generated by $X$ (Voiculescu), this is also true of 1-boundedness (Jung, E.).
- The character of the fundamental representation is semicircular (Banica), thus we get the theorem.
- Originally, the argument relied on the computation of the spectral measures of the generators by Banica–Collins–Zinn-Justin.
- This strategy can also be adapted to prove that $L^\infty(O_{2M}^{J,+})$ are also strongly 1-bounded (E.).

# More on Free Entropy Dimension and CQGs

- Let $\mathbb{G}$ be a compact quantum group with defining representation $v$ such that $L^\infty(\mathbb{G})$ is a $\mathrm{II}_1$-factor.
- Then (Connes–Shlyakhtenko) $\delta_0(v) \leq 1 - \beta_0^{(2)}(\mathbb{G}) + \beta_1^{(2)}(\mathbb{G})$
- and $1 \leq \delta_0(v)$ if $L^\infty(\mathbb{G})$ is Connes embeddable (Jung).
- Thus $\delta_0(v) = 1$ for $\mathbb{G} = O_N^+, O_{2M}^{J,+}, \mathrm{Aut}^+(B)$ (!) and $1 \leq \delta_0(v) \leq 2$ for $\mathbb{G} = U_N^+$ (many hands).
- We already know that the first two families give S1B $\mathrm{II}_1$-factors, we now show the same for $\mathrm{Aut}^+(B)$ when $dim(B) = N^2$ with $N \geq 3$, and that $U_N^+$ is NOT strongly 1-bounded.
- To the best of our knowledge, this gives the first proof that $L^\infty(U_N^+)$ is never isomorphic to $L^\infty(O_M^+)$ (other than Banica's result that $L^\infty(U_2^+) \cong \mathcal{L}\mathbb{F}_2$).

# Free Entropy Dimension and Subfactors

- Let $\mathcal{N} \subset \mathcal{M}$ be a unital inclusion of $\mathrm{II}_1$-factors; it can be assigned an *index* $[\mathcal{M} : \mathcal{N}]$ (Jones) which we will assume is finite.
- Suppose that $X, Y$ generates $\mathcal{N}, \mathcal{M}$ respectively, then one expects (Schreier)

$$\big(\delta_0(X) - 1\big) = [\mathcal{M} : \mathcal{N}]\,\big(\delta_0(Y) - 1\big)$$

**Theorem (Brannan–E.–Harris–Yamashita)**

*$\mathcal{N}$ is strongly 1-bounded if and only if $\mathcal{M}$ is strongly 1-bounded.*

- Hence $\mathrm{Aut}^+(M_N(\mathbb{C}))$ for $N \geq 3$ produces a S1B function algebra, as it appears as an index 2 subfactor of $L^\infty(O_N^+)$ (Banica, Brannan).
- By graded twisting, one can realize $L^\infty(O_{2M}^{J,+})$ as a finite index subfactor of $M_2(\mathbb{C}) \otimes L^\infty(O_{2M}^+)$ (Bichon–Neshveyev–Yamashita).

# Quantum Automorphism Groups as Subfactors

**Theorem (BEHY)**

*For any finite dimensional $C^*$-algebra $B$, we have a finite index inclusion of $L^\infty(\mathrm{Aut}^+(B))$ into a matrix amplification of $L^\infty(S^+_{\dim(B)})$.*

**Corollary**

*If $\dim(B) = N^2$ with $N \geq 3$, then $L^\infty(\mathrm{Aut}^+(B))$ is strongly 1-bounded. In particular $L^\infty(S^+_{N^2})$ is strongly 1-bounded.*

- For $B_1, B_2$, the associated quantum automorphism groups are monoidally equivalent if and only if $\dim(B_1) = \dim(B_2)$.
- These embeddings come from the *linking algebra* which implements the monoidal equivalence.
- This seems to be the main obstacle to pushing our results to arbitrary $\dim(B)$.

# The Free Unitary Case

**Proposition (Brown–Dykema–Jung)**

*Assume that $\mathcal{N}$ and $\mathcal{M}$ are Connes embeddable strongly 1-bounded $II_1$-factors, and that $X, Y$ generates $\mathcal{N}, \mathcal{M}$ respectively. Then in $\mathcal{N} * \mathcal{M}$ it holds that $\delta_0(X \cup Y) = 2$, and so $\mathcal{N} * \mathcal{M}$ is not strongly 1-bounded.*

- One can realize $L^\infty(U_N^+)$ as a graded twist of $L^\infty(O_N^+ * O_N^+)$, and hence as a finite index subfactor of a matrix amplification. Therefore it cannot be strongly 1-bounded.
- In particular, we can forbid the canonical generators from forming a 1-bounded set.
- However, at the moment we cannot yet prove that $\delta_0(v)$ is different from 1 (we expect it to equal 2).
- While $L^\infty(\mathrm{Aut}^+(M_N(\mathbb{C})))$ also embeds into $L^\infty(U_N^+)$, this can never have finite index.

# Proof Sketch of the Subfactor Result I

- From now on let $\mathcal{N} \subset \mathcal{M}$ be a unital finite index inclusion and assume that $\mathcal{M}$ is strongly 1-bounded.
- The *basic construction* gives a new $\mathrm{II}_1$-factor $\mathcal{M}_{\mathrm{ov}} \supset \mathcal{M}$ such that $[\mathcal{M}_{\mathrm{ov}} : \mathcal{M}] = [\mathcal{M} : \mathcal{N}]$.
- In fact, $\mathcal{M}_{\mathrm{ov}}$ is generated by $\mathcal{M}$ and the *Jones projection* $e \in B(L^2(M))$, which commutes with $N$.
- Moreover, $\mathcal{M}_{\mathrm{ov}}$ is an amplification of $\mathcal{N}$, since $\mathcal{M}_{\mathrm{ov}}$ and $\mathcal{N}^{op}$ are commutants on $L^2(\mathcal{M})$.
- Hayes: corners of a strongly 1-bounded $\mathrm{II}_1$-factor with respect to a projection $p$ are strongly 1-bounded if $\tau(p) \neq 0$.
- So it suffices to show that $\mathcal{M}_{\mathrm{ov}}$ is strongly 1-bounded.

# Proof Sketch of the Subfactor Result II

**Theorem (Jung)**

*Let $\mathcal{A} \subset \mathcal{B}$ be a unital inclusion of $II_1$-factors with $\mathcal{A}$ strongly 1-bounded. Assume that we have a unitary $u \in \mathcal{B}$ such that there is a diffuse self-adjoint element $a \in \mathcal{A}$ such that $uau^* \in \mathcal{A}$. Then the von Neumann algebra generated by $\mathcal{A}$ and $u$ is strongly 1-bounded.*

- Consider the self-adjoint unitary $1 - 2e$ and let $x$ be any diffuse self-adjoint element in $\mathcal{N}$, then

$$(1-2e)x(1-2e) = x - 2(ex + xe) + 4exe = x \in \mathcal{M}.$$

- Hence $\mathcal{M}_{\mathrm{ov}}$ is strongly 1-bounded.
- For the other direction, one uses instead the *tunnel construction*, which gives a subfactor $\mathcal{N}_{\mathrm{un}} \subset \mathcal{N}$ and a projection $f \in B(L^2(\mathcal{N}))$ commuting with $\mathcal{N}_{\mathrm{un}}$ such that $\mathcal{M}$ is generated by $\mathcal{N}$ and $f$.

# Outlook

- Can we prove S1Bness for the other quantum automorphism group function algebras?
- Can we determine $\delta_0(v)$ for $U_N^+$?
- What else can we push through these embeddings?
  - Certainly Connes Embeddability and RFDness
  - What about inner linearity for instance?
- What about properties for the linking algebra lurking in the background? (relation to quantum information theory)

UiO : Department of Mathematics
University of Oslo

Floris Elzinga

# Strongly 1-Bounded Quantum Group von Neumann Algebras